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LETTER TO THE EDITOR

Lorenz model for the rotating Rayleigh–Bénard problem

J K Bhattacharjee† and A J McKane‡

† Department of Physics§, Indian Institute of Technology, Kanpur 208016, India and Department of Physics, University of Manchester, Manchester M13 9PL, UK
‡ Department of Theoretical Physics, University of Manchester, Manchester M13 9PL, UK

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Abstract. A four-mode Lorenz model for the rotating Rayleigh-Bénard system is constructed which reproduces the results of a linear stability analysis of the hydrodynamic equations. An extension of this model predicts that the Küppers-Lortz instability occurs at a critical Taylor number of $T_c = 80\pi^4$ and a critical angle θ_c given by $\tan\theta_c = 4/\sqrt{5}$.

An interesting hydrodynamic instability was discovered by Küppers and Lortz [1, 2] who showed that in a rotating Rayleigh-Bénard system, if the rotation speed exceeds a critical value, the two-dimensional roll solutions are no longer stable against perturbations by rolls with a different axis. The critical dimensionless rotation speed was found to be $\Omega_c \simeq 24$ for large Prandtl number, σ, and free boundary conditions. It was shown by Busse and Clever [3], from a numerical analysis, that once the critical speed was exceeded there was evidence of noise in the system and this direct transition to turbulence at the Küppers-Lortz threshold was experimentally observed by Niemela and Donnelly [4]. In this letter we propose a Lorenz model for the rotating Rayleigh-Bénard problem that not only reproduces the thresholds of convection (stationary and oscillatory) correctly for free boundaries but also exhibits the Küppers-Lortz instability. In fact the favoured angle for the off-axis perturbing rolls and the critical rotation speed can be analytically determined in closed form.

The hydrodynamic equations describing the rotating system require three independent variables in the Boussinesq approximation. These are the z component of the velocity field, the z component of the vorticity and the fluctuation of the temperature from the steady state profile in the absence of convection. We use dimensionless variables, scaling distances by d (plate separation), time by d^2/ν (ν is the kinematic viscosity), temperature by ΔT (temperature difference between the plates) and velocity by λ/d (λ is the thermal diffusivity). The dimensionless variables are w (z component of velocity), ζ (z component of vorticity) and θ (the temperature fluctuations). The governing equations are [5]

$$\nabla^2\left(\nabla^2 - \frac{\partial}{\partial t}\right)w = \tau D\zeta - R\nabla_2^2\theta \tag{1}$$

$$\left(\nabla^2 - \frac{\partial}{\partial t}\right)\zeta = -\tau D w \tag{2}$$

$$\left(\nabla^2 - \sigma\frac{\partial}{\partial t}\right)\theta = -w + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\theta \tag{3}$$

§ Permanent address.

where we have dropped the non-linear terms in the equations for w and ζ (anticipating the use of free boundary conditions which do not lead to contributions from these terms in the Lorenz model), R is the Rayleigh number, $T=\tau^2=4\Omega^2 d^4/\nu^2$ is the Taylor number and D is the operator $\partial/\partial z$.

The linear stability analysis of the above system shows that stationary convection occurs at

$$R_c=\frac{(\pi^2+a_c^2)^3+T\pi^2}{a_c^2} \tag{4}$$

where

$$2\left(\frac{a_c}{\pi}\right)^6+3\left(\frac{a_c}{\pi}\right)^4=1+\frac{T}{\pi^4}. \tag{5}$$

The standard Lorenz model for two-dimensional rolls can be obtained by the following expansions for the different fields (the roll axis is taken to be the y axis)

$$\omega=a(t)\cos a_c x\sin \pi z \tag{6}$$

$$\zeta=f(t)\cos a_c x\cos \pi z \tag{7}$$

$$\theta=b(t)\cos a_c x\sin \pi z+c(t)\sin 2\pi z. \tag{8}$$

Inserting the above expansions into (1)-(3) and equating coefficients of Fourier terms on either side we arrive at

$$\dot{X}=\sigma(-X+Y+tG) \tag{9a}$$

$$\dot{G}=-\sigma(G+tX) \tag{9b}$$

$$\dot{Y}=-XZ+rX-Y \tag{9c}$$

$$\dot{Z}=XY-bZ \tag{9d}$$

where X, Y, Z and G are scaled versions of a, b, c and f respectively, $r=R/R_c$, $t=\pi\tau/(\pi^2+a_c^2)^{3/2}$, $\tau=T^{1/2}$ and $b=4\pi^2/(a_c^2+\pi^2)$. The state $X=Y=Z=G=0$ is the conduction state which is destabilised in favour of stationary convection when $r=1+t^2$ in agreement with (4). The model also allows for oscillatory convection and the results obtained from this system are in agreement with the linear stability analysis of the full hydrodynamic equations with free boundary conditions. The effect of the non-linear terms can also be treated and will be discussed in detail elsewhere.

Here we wish to extend the above model to include the Küppers-Lortz instability. To do so we need to introduce y dependence in the fields and accordingly we take

$$w=a(t)\cos a_c x\sin \pi z+a_1(t)\cos(k_1x+k_2y)\sin \pi z \tag{10}$$

$$\zeta=f(t)\cos a_c x\cos \pi z+f_1(t)\cos(k_1x+k_2y)\cos \pi z \tag{11}$$

$$\theta=b(t)\cos a_c x\sin \pi z+c(t)\sin 2\pi z+b_1(t)\cos(k_1x+k_2y)\sin \pi z$$
$$+[c_1(t)\cos(k_1x+a_cx+k_2y)+c_2(t)\cos(k_1x-a_cx+k_2y)]\sin 2\pi z. \tag{12}$$

The corresponding Lorenz model is

$$\dot{X}=\sigma(-X+Y+tG) \tag{13a}$$

$$\dot{G}=-\sigma(G+tX) \tag{13b}$$

$$\dot{Y} = -XZ + rX - Y - X_1Z_1\frac{1-\cos\theta}{4} - X_1Z_2\frac{1+\cos\theta}{4} - G_1Z_1c\frac{\sin\theta}{4} + G_1Z_2c\frac{\sin\theta}{4} \quad (13c)$$

$$\dot{Z} = XY + X_1Y_1 - bZ \quad (13d)$$

$$\dot{X}_1 = \sigma(-X_1 + Y_1 + tG_1) \quad (13e)$$

$$\dot{G}_1 = -\sigma(G_1 + tX_1) \quad (13f)$$

$$\dot{Y}_1 = -X_1Z + rX_1 - Y_1 - XZ_1\frac{1-\cos\theta}{4} - XZ_2\frac{1+\cos\theta}{4} + c\frac{\sin\theta}{4}GZ_1 - c\frac{\sin\theta}{4}GZ_2 \quad (13g)$$

$$\dot{Z}_1 = \frac{(1-\cos\theta)}{2}(YX_1 + XY_1) - b_+Z_1 + c\frac{\sin\theta}{2}(G_1Y - GY_1) \quad (13h)$$

$$\dot{Z}_2 = \frac{(1+\cos\theta)}{2}(YX_1 + XY_1) - b_-Z_2 + c\frac{\sin\theta}{2}(GY_1 - G_1Y). \quad (13i)$$

Here $k_1^2 + k_2^2 = a_c^2$, $k_1 = a_c\cos\theta$, $k_2 = a_c\sin\theta$, $c = (\pi^2 + a_c^2)^{1/2}/\pi$ and X_1, Y_1, Z_1, Z_2 and G_1 are scaled forms of a_1, b_1, c_1, c_2 and f_1 respectively and

$$b_\pm = \frac{4\pi^2 + 2a_c^2(1\pm\cos\theta)}{\pi^2 + a_c^2}.$$

It is easy to see that the trivial fixed point describing the conduction state is $X = Y = Z = G = X_1 = Y_1 = Z_1 = Z_2 = G_1 = 0$. The two-dimensional rolls are given either by $X_1 = Y_1 = Z_1 = Z_2 = G_1 = 0$ and the others non-zero or by $X = Y = G = Z_1 = Z_2 = 0$ and the remaining ones non-zero. The Küppers–Lortz instability pertains to the perturbing effect of one of the roll systems on the other. Supposing the roll system is formed with the axis in the y direction, we ask the question whether it can be destabilised by the other set of rolls whose axis is not along the y axis. The rolls along the y axis are described by

$$Z_0 = r - (t^2 + 1) \quad (14a)$$

$$G_0 = -tX_0 \quad (14b)$$

$$Y_0 = X_0(1 + t^2) \quad (14c)$$

and

$$X_0^2 = (1 + t^2)^{-1}bZ_0. \quad (14d)$$

The linear stability of this state against perturbation by X_1, Y_1, G_1, Z_1 and Z_2 can be studied from a linearised five by five system.

Trying solutions of the form e^{pt}, we find that $\operatorname{Re} p = \operatorname{Im} p = 0$ if

$$\tau\sin\theta = (\pi^2 + a_c^2)\frac{b_+(1+\cos\theta)^2 + b_-(1-\cos\theta)^2}{b_+(1+\cos\theta) - b_-(1-\cos\theta)}. \quad (15)$$

We can solve for θ in terms of τ and a_c from the above to find

$$\tan\theta = (x+2)^{-1}[\Gamma \pm \sqrt{\Gamma^2 - 4(x+1)(x+2)}] \quad (16)$$

where $x = a_c^2/\pi^2$ and $\Gamma = \tau/\pi^2$. Clearly the minimum value of τ will be obtained for

$$\tau^2 = 4(x+1)(x+2)\pi^4 \quad (17)$$

which, combined with (5) leads to $x = 3$, implying that

$$\tan\theta_c = \sqrt{\tfrac{16}{5}} \qquad \theta_c \simeq 61° \tag{18}$$

and

$$\tau_c = \sqrt{80}\,\pi^2. \tag{19}$$

Thus, the critical rotation speed above which the two-dimensional rolls are unstable is found to be somewhat higher than the result of Küppers and Lortz, although the critical angle for the axis roll agrees quite well. The non-trivial fixed point describing the simultaneous presence of the two sets of rolls can be easily obtained from (13a)-(13i) and is found to be

$$Y = X(1+t^2),\ G = -tX \tag{20a}$$

$$Y_1 = X_1(1+t^2),\ G_1 = -tX_1 \tag{20b}$$

$$b_+Z_1 = (1-\cos\theta)(1+t^2)XX_1 \tag{20c}$$

$$b_-Z_2 = (1+\cos\theta)(1+t^2)XX_1 \tag{20d}$$

$$bZ = (1+t^2)(X^2+X_1^2) \tag{20e}$$

$$\Delta X^2 = \Delta_1 X_1^2 \tag{20f}$$

where

$$\Delta = ct\sin\theta[b_+(1+\cos\theta) - b_-(1-\cos\theta)] - [b_+(1+\cos\theta)^2 + b_-(1-\cos\theta)^2] \tag{21a}$$

$$\Delta_1 = ct\sin\theta[b_-(1-\cos\theta) - b_+(1+\cos\theta)] - [b_+(1+\cos\theta)^2 + b_-(1-\cos\theta)^2] \tag{21b}$$

and

$$X^2\left[\frac{1}{b}\left(1+\frac{\Delta}{\Delta_1}\right) - \frac{\Delta}{4b_+b_-}\right] = \frac{r}{1+t^2} - 1. \tag{22}$$

We note that Δ_1 is always less than zero. Thus X_1 can exist only if Δ is less than zero, i.e. for $\tau < \tau_c$. For $\tau > \tau_c$, the roll system with axis at an angle to the y axis cannot exist as a steady state, and the roll system with axis along the y axis is unstable to perturbations by the other set of rolls. Since the destabilisation of the two-dimensional rolls at $\tau = \tau_c$ does not occur via a Hopf bifurcation, it is not expected to lead to a simple oscillatory state. Consequently we conjecture that for $\tau > \tau_c$ a direct transition to a chaotic state occurs at $r = 1 + t^2$.

We end by noting that our value of τ_c is significantly different from that obtained by Küppers and Lortz. The reason lies in the fact that the amplitudes a_1, b_1, f_1, c, c_1 and c_2 are all independent variables in the Lorenz model, whereas in the analysis of the hydrodynamic equations these coefficients are obtained from a perturbative solution of (1)-(3).

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